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CYLINDRICAL BENDING OF THICK PLATES

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Abstract—Sixth and twelfth order plate theory solutions for cylindrical bending plate problems, efficiently obtained by a preliminary reduction of the twelfth order theory for general cylindrical bending to a boundary value problem for a fourth order ODE, demonstrate the adequacy of these theories for the interior of thick plates with a thickness-to-span ratio λ up to 1/2 (at least). In contrast, Kirchhoff's classical thin plate theory is not adequate for values of the same ratio greater than 1/10. Comparison with the corresponding interior solutions of 3-D elastostatics shows that the twelfth order theory captures the (numerically small) first order correction term in the thickness-to-span parameter ε not possible by lower order plate theories. While it is a qualitatively important feature, this additional term associated with the effect of transverse normal stress does not significantly improve the plate theory approximation quantitatively for isotropic plates.

1. INTRODUCTION

This note is concerned with the small deformation of a homogeneous, isotropic, elastic flat plate of uniform thickness $2h$. Under external loads, the three-dimensional elastostatic response of a flat plate is now known to consist of an *interior* component significant throughout the plate and a *boundary layer* component which is negligibly small except in a narrow region adjacent to the edge(s) of the plate, Gregory (1991). A method was developed recently by Gregory and Wan (1984, 1985a,b, 1988) for obtaining the interior solution component, up to terms exponentially small in the thickness parameter, without any reference to the boundary layer solution component. The interior solution of several boundary value problems were solved in Gregory and Wan (1984, 1985b, 1988) and in Lin and Wan (1990a,b) to illustrate the application of this method. The results show that the corresponding Kirchhoff's thin plate solutions generally deviate from these interior solutions by more than 35% for a thickness-to-span ratio h/L equal to 0.2 (and by more than 10% for $h/L = 0.1$).

To the extent that a two-dimensional plate theory is often more practical (than the interior solution method of Gregory and Wan) for the analysis of complex plate problems, a more refined plate theory (than the Kirchhoff theory) is needed for plates with $h/L > 0.1$. In this paper, we use the cylindrical bending problems of Gregory and Wan (1984) to obtain some indication of the range of applicability of the sixth order theory of Reissner (1945) and the twelfth order theory of Lo *et al.* (1977) and Reissner (1983, 1987). Our analysis is simplified considerably by a preliminary reduction of the twelfth order theory for cylindrical bending to a two-point boundary value problem (BVP) for a fourth order ODE. The solution of this BVP shows that the twelfth order theory actually captures new qualitative features of the exact solution of the three-dimensional problem not already contained in the solution of the lower order plate theories. At the same time, we find that the sixth order theory is quantitatively adequate for $h/L \leqslant 1/2$ and very accurate for $h/L \leqslant 0.2$.

2. CYLINDRICAL BENDING UNDER UNIFORM PRESSURE

The problem of interest here is an infinitely long isotropic, homogeneous, linearly elastic plate strip free of interior loading and under uniform pressure $\pm q/2$ at the two plate faces $x_3 = \pm h$. At the two edges of the plate strip $x_1 = \pm L$, the plate is fixed so that the displacement components vanish there:

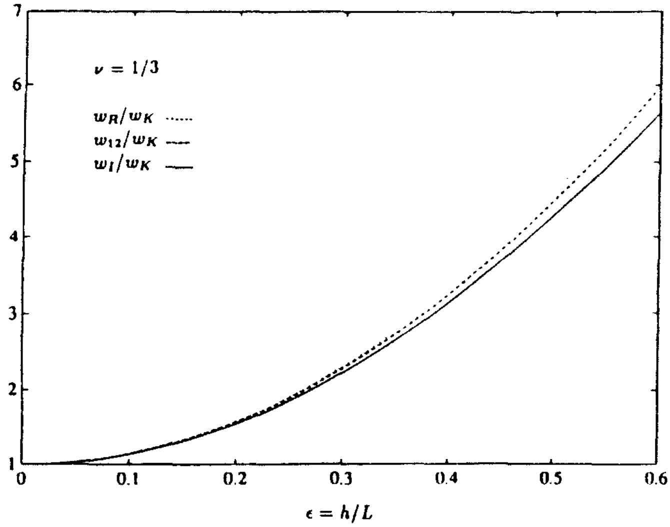

Fig. 1. Midplane center displacements for $\nu = 1/3$ (normalized by the corresponding Kirchhoff theory solution) as functions of the thickness-to-span ratio $\varepsilon = h/L$ obtained from (i) the sixth order plate theory, (ii) the twelfth order plate theory, and (iii) the interior solution of Gregory and Wan.

$$x_1 = \pm L: \quad u_1 = u_2 = u_3 = 0 \quad (|x_2| < \infty, \quad |x_3| \leqslant h).$$

The linear elastostatic response of such a plate to the prescribed uniform pressure distributions is a state of plane strain with all stress and displacement components uniform in the x_2 direction. The interior solution for this problem has been obtained in Gregory and Wan (1984). The transverse displacement at the center of the plate is given by that solution up to exponentially small terms in $\varepsilon = h/L$. We denote this solution by $w_1 \equiv u_3^1(0,0)$ with

$$\frac{w_1}{w_K} = \frac{(1-\nu)+\nu t_2^F \varepsilon^2}{2(1-\nu)-\nu t_2^B \varepsilon}\left\{4 - \frac{6\nu}{1-\nu} t_2^B \varepsilon + \frac{4\varepsilon^2}{1-\nu}[(2-\nu)n_3^B + 3\nu] - \left[6t_2^B - \frac{1+\nu}{1-\nu} t_4^B\right]\varepsilon^3\right\}$$
$$- \left\{1 - \frac{\varepsilon^2}{1-\nu}[12-6\nu-6\nu t_2^F] - \frac{4\varepsilon^3}{1-\nu}(2-\nu)n_3^F - \left[\frac{1+\nu}{1-\nu} t_4^F - 6t_2^F\right]\varepsilon^4\right\}, \quad (1a)$$

where

$$w_K = \frac{qh(1-\nu^2)}{16E\varepsilon^4} \equiv \frac{qL^4}{4!\,D_M} \quad (1b)$$

is the corresponding Kirchhoff plate theory solution. The coefficients $\{t_i^B, t_i^F\}$ depend only on Poisson's ratios ν and are given in Gregory and Wan (1984) for $\nu = 1/4$, $1/3$ and $1/2$.

The expression (1) for w_1/w_K reduces to unity for $\varepsilon = h/L = 0$. For $\varepsilon > 0$, we intuitively expect w_1 to be larger than w_K as a Kirchhoff plate is effectively transversely rigid (with all transverse shear and transverse normal strains vanishing identically). We learn from the results of Gregory and Wan (1984) that this is in fact the case except for $\varepsilon < 10^{-2}$ since t_2^B was found to be $0(10^{-1})$ there. The dependence of the center displacement on the aspect ratio given by (1) can be found in Gregory and Wan (1984) for $\nu = 1/2$ and is shown in Fig. 1 for $\nu = 1/3$. We see that w_1 is one and one-half times that of the Kirchhoff theory for $\varepsilon = 1/5$. In general, the interior solution provides a useful bench mark for the approximate solutions by higher order plate theories.†

† There are a number of printing errors and inconsistencies in the results of Gregory and Wan (1984) which have been corrected above. For example, the very last term of eqn (8.11) in Gregory and Wan (1984) (multiplying ε^4) should be $6(\nu-1)t_2^F$ instead of $6\nu t_2^F$; this has been corrected in (1). The correction involves a change of order $(h/L)^4$ and is therefore negligibly small for $\varepsilon \leqslant 0.2$. Also, the thickness parameter H was set equal to 1 in Section 8 of Gregory and Wan (1984).

3. THE BOUNDARY VALUE PROBLEM FOR THE TWELFTH ORDER PLATE THEORY

The method for the correct interior solution for plate problems developed in Gregory and Wan (1984, 1985a, 1988) requires a substantial amount of calculation even for simple problems such as cylindrical bending. It is often preferable to obtain more tractable, accurate two-dimensional plate theory solutions for the same problems. For the more refined twelfth order plate theory of Reissner (1983, 1987), the stress distributions in the deformed body are stipulated to be

$$(\sigma_{11},\sigma_{22},\sigma_{12}=\sigma_{21})=\frac{3}{2}\frac{(M_{11},M_{22},M_{12}=M_{21})}{h^2}\frac{x_3}{h}+\frac{3}{2}\frac{(P_{11},P_{22},P_{12}=P_{21})}{h^2}\left(\frac{x_3}{h}-\frac{5}{3}\frac{x_3^3}{h^3}\right),$$

$$(\sigma_{13}=\sigma_{31},\sigma_{23}=\sigma_{32})=\frac{3}{4}\frac{(Q_1,Q_2)}{h^2}\left(1-\frac{x_3^2}{h^2}\right)+\frac{1}{8}\frac{(S_1,S_2)}{h}\left(1-6\frac{x_3^2}{h^2}+5\frac{x_3^4}{h^4}\right),$$

$$\sigma_{33}=-\frac{3}{4}q\left(\frac{x_3}{h}-\frac{1}{3}\frac{x_3^3}{h^3}\right)-\frac{1}{8}T\left(\frac{x_3}{h}-2\frac{x_3^3}{h^3}+\frac{x_3^5}{h^5}\right). \tag{2a}$$

The two-dimensional stress measures $M_{ij}(x_1,x_2)$, $Q_j(x_1,x_2)$, $P_{ij}(x_1,x_2)$, $S_j(x_1,x_2)$ and $T(x_1,x_2)$ are to be determined by the two-dimensional equilibrium equations

$$Q_{1,1}+Q_{2,2}+q=0,\qquad S_{1,1}+S_{2,2}-T=0$$

$$M_{1j,1}+M_{2j,2}-Q_j=0,\quad P_{1j,1}+P_{2j,2}-S_j=0\quad (j=1,2) \tag{2b}$$

and the Euler differential equations of the variational principle for stresses and displacements with (2a) as the comparison functions, Reissner (1983). The following weighted average of the displacement components appear naturally in the resulting two-dimensional plate theory:

$$(w,v)=\int_{-h}^{h}\frac{u_3}{h}\left[\left(\frac{3}{4}-\frac{3}{4}\frac{x_3^2}{h^2}\right),\left(\frac{1}{8}-\frac{3}{4}\frac{x_3^2}{h^2}+\frac{5}{8}\frac{x_3^4}{h^4}\right)\right]\mathrm{d}x_3$$

$$(\phi_1,\phi_2)=\int_{-h}^{h}\frac{(u_1,u_2)}{h^2}\frac{3x_3}{2h}\,\mathrm{d}x_3,\quad (\psi_1,\psi_2)=\int_{-h}^{h}\frac{(u_1,u_2)}{h^2}\left(\frac{3}{2}\frac{x_3}{h}-\frac{5}{2}\frac{x_3^3}{h^3}\right)\mathrm{d}x_3. \tag{2c}$$

For cylindrical plate bending problems, we have $(\ )_{,2}\equiv 0$ and four of the equilibrium equations for the twelfth order theory in (2b) simplify to

$$Q_{1,1}=-q,\quad M_{11,1}=Q_1,\quad S_{1,1}=T,\quad P_{1,1}=S_1. \tag{3}$$

The corresponding linear elastic stress–strain relations for a transversely isotropic plate from Reissner (1983) are

$$M_{11}=D_M[\phi_{1,1}+(1+\nu)(B_{Mq}q+B_{MT}T)], \tag{4a}$$

$$M_{22}=D_M[\nu\phi_{1,1}+(1+\nu)(B_{Mq}q+B_{MT}T)] \tag{4b}$$

$$\phi_1+w_{,1}=B_QQ_1+B_{QS}S_1,\quad \psi_1+v_{,1}=B_SS_1+B_{QS}Q_1 \tag{5a, b}$$

$$P_{11}=D_P[\psi_{1,1}+(1+\nu)(B_{Pq}q+B_{PT}T)], \tag{6a}$$

$$P_{22}=D_P[\nu\psi_{1,1}+(1+\nu)(B_{Pq}q+B_{PT}T)] \tag{6b}$$

$$v=C_{Tq}q+C_TT-B_{MT}(M_{11}+M_{22})-B_{PT}(P_{11}+P_{22}) \tag{7}$$

where

$$B_Q = 21B_{QS} = \frac{189}{4}B_S = \frac{3}{5Gh}, \quad C_q = \frac{1683}{2}C_T = \frac{153}{2}C_{Tq} = \frac{34h}{35E_z}$$

$$D_M = \frac{4}{21}D_P = \frac{2Eh^3}{3(1-\nu^2)}, \quad B_{Mq} = 21B_{MT} = 21B_{Pq} = \frac{189}{4}B_{PT} = \frac{3\nu_z}{5h\sqrt{EE_z}}. \tag{8}$$

For isotropic plates, we have $E_z = E$, $\nu_z = \nu$ and $G = E/2(1+\nu)$.

The two edges $x_1 = \pm L$ of the plate for our cylindrical bending problems are clamped so that

$$x_1 = \pm L: \quad w = \phi_1 = \psi_1 = v = 0. \tag{9}$$

For a homogeneous plate of constant thickness, the quantities w, M_{11}, M_{22}, v, P_{11}, T and P_{22} are seen to be even functions of x_1 and Q_1, ϕ_1, S_1 and ψ_1 are odd functions of x_1. Therefore, it suffices to obtain the solution for $0 \leqslant x_1 \leqslant L$ subject to the end conditions

$$x_1 = 0: \quad \phi_1 = \psi_1 = v_{,1} = w_{,1} = 0, \tag{10a}$$

$$x_1 = L: \quad \phi_1 = \psi_1 = v = w = 0. \tag{10b}$$

The differential equations (3), (4a), (5a), (5b) and (6a) constitute an eighth order linear system. The eight boundary conditions (9) (or (10a) and (10b)) are consistent with the order of this system. The first two equilibrium equations in (3) can be integrated immediately to give

$$Q_1 = -qx_1, \quad M_{11} = M_0 - \tfrac{1}{2}qx_1^2 \tag{11, 12}$$

where M_0 is an unknown constant of integration to be determined by the end conditions. We have made use of the fact that Q_1 is an odd function to eliminate a second constant of integration.

4. SOLUTION OF THE SIXTH ORDER THEORY

Reissner's sixth order plate theory is obtained from the twelfth order theory by setting $S_i = P_{ij} = T = 0$ in (2) (and in the variational functional used in Reissner (1983)). Introduction of the expression for M_{11} in (12) into (4a) and integrating the result gives

$$D_M\phi_1 = M_0x_1 - \tfrac{1}{6}qx_1^3 - (1+\nu)D_MB_{Mq}qx_1 \tag{13}$$

where we have used the fact that ϕ_1 is an odd function to eliminate the constant of integration. The edge condition $\phi_1(L) = 0$ determines M_0 to be

$$M_0 = \tfrac{1}{6}qL^2 + (1+\nu)D_MB_{Mq}q = \frac{qL^2}{6}\left\{1 + 6(1+\nu)\frac{D_MB_{Mq}}{L^2}\right\}. \tag{14}$$

With ϕ_1 completely determined, the relation (5a) becomes an equation for $w_{,1}$ which can be integrated to give

$$D_Mw = D_Mw_0 - \tfrac{1}{2}M_0x_1^2 + \tfrac{1}{24}qx_1^4 + \tfrac{1}{2}qx_1^2D_M[(1+\nu)B_{Mq} - B_Q] \tag{15}$$

where w_0 is a constant of integration and M_0 is given by (14). Just as in Kirchhoff's theory, the solution (15) is smooth throughout the plate; it contains no layer components near the plate edges.

The edge condition $w(L) = 0$ determines the constant w_0 which gives the transverse displacement at the isotropic plate center (w_R) of Reissner's sixth order plate theory:

$$\frac{w_R}{w_K} = 1+\frac{12D_M B_Q}{L^2} = 1+\frac{48}{5(1-\nu)}\varepsilon^2, \quad \varepsilon = \frac{h}{L}. \tag{16}$$

Not surprisingly, w_R tends to the Kirchhoff solution as $\varepsilon \to 0$. For $h/L > 0$, the more flexible Reissner plate (which allows for transverse shear deformations) suffers a larger center deflection as expected. The graphs in Fig. 1 show that w_R is also larger than the corresponding interior solution value w_I. Given that the sixth order theory is a consequence of the principle of minimum complementary energy for a restricted class of comparison functions, this is also not surprising. Numerical results for $\nu = 1/3$ show that w_R exceeds the interior solution w_I by less than 7.3% for $\varepsilon = 1/2$ and less than 2.4% for $\varepsilon = 0.2$; the corresponding Kirchhoff solution is only two thirds of the interior solution for $\varepsilon = 0.2$!

While the sixth order theory is sufficiently accurate for an approximate solution, its first order correction to the Kirchhoff solution is $0(\varepsilon^2)$ and not $0(\varepsilon)$ as shown in (1). Though the $0(\varepsilon^2)$ terms are in fact dominant except for $h/L < 10^{-2}$, it is still of theoretical interest to examine the first order correction of the twelfth order theory.

5. REDUCTION OF THE TWELFTH ORDER THEORY TO A TWO-POINT BVP

The first two equilibrium equations in (3) have already been integrated to give Q_1 and $M_{11} \equiv qL^2M$ up to a constant of integration M_0 (see (11) and (12)). The other two equilibrium equations can be used to express S_1 and T in terms of $P_{11} \equiv qL^2P$:

$$S_1 = qL^2P_{,1}, \quad T = S_{1,1} = qL^2P_{,11}. \tag{17a, b}$$

Equations (4a) and (6a) are then used to express $\phi_{1,1}$ and $\psi_{1,1}$ in terms of P and known quantities:

$$D_M\phi_{1,1} = qL^2\left\{M-\frac{D_M(1+\nu)}{L^2}[B_{Mq}+B_{MT}P''(x)]\right\} \tag{18a}$$

$$D_P\psi_{1,1} = qL^2\left\{P-\frac{D_P(1+\nu)}{L^2}[B_{Pq}+B_{PT}P''(x)]\right\} \tag{18b}$$

where $x = x_1/L$ and $(\ )' = \mathrm{d}(\ )/\mathrm{d}x$. These expressions allow us to eliminate $\phi_{1,1}$ and $\psi_{1,1}$ from (4b), (6b) and (7). The resulting expression for v is

$$v = q\{\bar{C}_{Tq}+\bar{C}_T P''-(1+\nu)B_{MT}L^2M-(1+\nu)B_{PT}L^2P\}, \tag{18c}$$

where

$$\bar{C}_{Tq} = C_{Tq}-(1-\nu^2)D_M B_{Mq}B_{MT}-(1-\nu^2)D_P B_{Pq}B_{PT}, \tag{18d}$$

$$\bar{C}_T = C_T-1(1-\nu^2)D_M B_{MT}^2-(1-\nu^2)D_P B_{PT}^2. \tag{18e}$$

The corresponding expressions for M_{22} and P_{22} will not be needed here.

Next, we use (3d) and (18c) to eliminate S_1 and v from (5b) to get ψ_1 in terms of P:

$$\psi_1 = -qL\left\{\frac{\bar{C}_T}{L^2}P'''-[B_S+(1+\nu)B_{PT}]P'+[B_{QS}+(1+\nu)B_{MT}]x\right\}. \tag{19}$$

Upon substituting (19) into (18b), we obtain a single fourth order ODE for P alone:

$$\frac{D_P\bar{C}_T}{L^4}P''''-\frac{D_P}{L^2}[B_S+2(1+\nu)B_{PT}]P''+P = -\frac{D_P B_{QS}}{L^2} \tag{20}$$

where we have made use of $B_{Pq} = B_{MT}$ to simplify the right-hand side.

The fourth order ODE (20) is supplemented by four boundary conditions. We observed earlier that P is an even function of x. Hence we must have

$$x = 0: \quad P' = P''' = 0. \tag{21a, b}$$

The condition (21a), $P'(0) = 0$, is also a consequence of (5b), (3d), (11) and the second and third boundary conditions in (10a). The other condition, $P'''(0) = 0$, can also be obtained from the boundary condition $\psi_1(0) = 0$ (see (10a)) along with (19) and (21a).

Two additional boundary conditions for P are obtained from the second and third boundary condition stipulated in (10b). Expressed in terms of P by (19), the condition $\psi_1(x_1 = L) = 0$ gives a third boundary condition on P:

$$x = 1: \quad \frac{\bar{C}_T}{L^2} P''' - [B_S + (1+\nu)B_{PT}]P' = -[B_{QS} + (1+\nu)B_{MT}]. \tag{22a}$$

The condition $v(x_1 = L) = 0$ expressed in terms of P by (18c) gives the fourth boundary condition. For the result to be useful, we still need to determine the unknown constant M_0 in $M(x)$. This is done with the help of the first boundary condition in (10b), $\phi_1(x_1 = L) = 0$. An expression for ϕ_1 in terms of $P(x)$ can be obtained by integrating both sides of (18a) and keeping in mind that $\phi_1(x)$ is an odd function. The result is

$$D_M\phi_1(x) = qL^3\left\{\frac{M_0}{qL^2}x - \frac{1}{6}x^3 - \frac{D_M(1+\nu)}{L^2}[B_{Mq}x + B_{MT}P'(x)]\right\}. \tag{23}$$

The condition $\phi_1(x_1 = L) = 0$ now gives

$$\frac{M_0}{qL^2} = \frac{1}{6} + \frac{D_M(1+\nu)}{L^2}[B_{Mq} + B_{MT}P'(1)]. \tag{24}$$

With the help of (24), we eliminate M_0 from (18c) and write the condition $v(x_1 = L) = 0$ as

$$\begin{aligned} x = 1: \quad & \bar{C}_T P'' - D_M B_{MT}^2(1+\nu)^2 P' - B_{PT}(1+\nu)L^2 P \\ & = -\bar{C}_{Tq} - (1+\nu)B_{MT}L^2\left[\frac{1}{3} - \frac{D_M B_{Mq}(1+\nu)}{L^2}\right]. \end{aligned} \tag{22b}$$

The ODE (20) and the four boundary conditions (21a, b) and (22a, b) define a two-point BVP for $P(x)$. Once we have obtained $P(x)$, we can calculate all the remaining stress and displacement quantities, except w, by way of (11), (12), (17a, b), (18c), (19), (23), (4b) and (6b).

6. THE CENTER DEFLECTION OF THE TWELFTH ORDER THEORY

To get the transverse deflection $w(x)$, we integrate (5a) to obtain

$$w = w_0 - \frac{qL^4}{D_M}\left\{\frac{x^2}{12} - \frac{x^4}{24} - \frac{D_M}{L^2}[B_{QS} + (1+\nu)B_{MT}]P + \frac{D_M}{L^2}[B_Q + (1+\nu)B_{MT}P'(1)]\frac{x^2}{2}\right\}. \tag{25}$$

The boundary condition $w(x_1 = L) = 0$ determines the constant w_0 and therefore the displacement at the plate center, denoted by w_{12}:

$$\frac{w_{12}}{w_K} = \frac{w_R}{w_K} + \frac{24D_M}{L^2}\left\{[B_{QS}+(1+\nu)B_{MT}][P(0)-P(1)] + \frac{1}{2}[(1+\nu)B_{MT}]P'(1)\right\}. \quad (26)$$

The terms of (26) in braces are expected to reduce w_R/w_K toward w_l/w_K, given that a larger class of trial solutions is used in the principle of minimum complementary energy to get the twelfth order theory.

7. TRANSVERSELY INEXTENSIBLE PLATES

For the special case of $\nu_z = 1/E_z = 0$, the BVP for $P(x)$ simplifies considerably. From (8), we have

$$C_q = C_{Tq} = C_T = 0, \quad B_{Mq} = B_{MT} = B_{Pq} = B_{PT} = 0. \quad (27, 28)$$

It follows from (27), (28) and (18c) that $\bar{C}_T = 0$ and the DE (20) is reduced to

$$\frac{D_P B_S}{L^2} P'' - P = \frac{D_P B_{QS}}{L^2}. \quad (29)$$

From (27), (28), (23) and (19), the condition $\phi_1(0) = 0$ in (10a) is trivially satisfied and the condition $\psi_1(0) = 0$ requires

$$P'(0) = 0. \quad (30a)$$

With (30a), we see from (18c) and (25) that the other two conditions in (10a) are also satisfied without any additional requirements on $P(x)$. In fact, we have $v(x_1) \equiv 0$ for this theory.

At the other end, the condition $\psi_1(x_1 = L) = 0$ requires

$$P'(l) = \frac{B_{QS}}{B_S} = \frac{1}{21}. \quad (30b)$$

The condition $v(x_1 = L)$ is again trivially satisfied while the other two conditions in (10b) serve to determine M_0 and w_0.

The exact solution of the BVP defined by (29) and (30a, b) is

$$P = \frac{\varepsilon}{21\alpha}\frac{\cosh(\alpha x/\varepsilon)}{\sinh(\alpha/\varepsilon)} - \frac{D_P B_{QS}}{L^2} \quad (31)$$

where $\alpha = \sqrt{45(1-\nu)/4}$ for a transversely inextensible material. If the plate is very thin so that $0 < \varepsilon \ll 1$, the solution (31) has a boundary layer component with a layer width of the order of h/L. Away from the edges, we have $P \sim -D_P B_{QS}/L^2$.

The corresponding center deflection, denoted by w_{10}, is obtained from (26):

$$\frac{w_{10}}{w_K} = \frac{w_R}{w_K} + \frac{24D_M B_{QS}}{L^2}[P(0) - P(1)]. \quad (32)$$

For $0 < \varepsilon \ll 1$, we have for an isotropic plate

$$P(0) \sim -\frac{D_P B_{QS}}{L^2} = -\frac{\varepsilon^2}{5(1-\nu)} < 0 \quad (33)$$

$$P(1) \sim \frac{\varepsilon}{21\alpha} - \frac{\varepsilon^2}{5(1-\alpha)} = \frac{\varepsilon}{3\sqrt{5(1-\nu)}} - \frac{\varepsilon^2}{5(1-\nu)} > 0. \quad (34)$$

It follows that $w_{10} < w_R$ for $\varepsilon > 0$. The correction to the center deflection from the sixth order theory solution, however, is of higher order, $0(\varepsilon^3)$ relative to the Kirchhoff solution. The numerical results for $w_{10}(> w_K)$ not shown in Fig. 1 confirm these observations (and also show that $w_{10} > w_{12}$).

8. SOLUTION BY THE TWELFTH ORDER PLATE THEORY

For an isotropic plate, the coefficient $D_P\bar{C}_T/L^4$ of P'''' in (20) is $0(\varepsilon^4)$ and the coefficient $D_P[B_S+2(1+v)B_{PT}]/L^2$ is $0(\varepsilon^2)$. Hence the ODE for P is of the singular perturbation type and we expect layer phenomena near the edges of the plate. With $-D_PB_{SQ}/L^2$ as an exact particular solution for homogeneous plates of constant thickness, the layer solutions come from the complementary solutions of the ODE.

The fourth order ODE (20) is of constant coefficient. It is straightforward to obtain the following general exact solution for this equation which is even in x:

$$P(x) = \frac{D_PB_{QS}}{L^2}\left[C_1\cosh\left(\frac{\mu_1}{\varepsilon}x\right)+C_2\cosh\left(\frac{\mu_2}{\varepsilon}x\right)-1\right] \tag{35}$$

where μ_1^2 and μ_2^2 are the two roots of

$$\frac{D_P\bar{C}_T}{L^4\varepsilon^4}\mu^4-\frac{D_P}{L^2\varepsilon^2}[B_S+2(1+v)B_{PT}]\mu^2+1 = 0 \tag{36}$$

(which are independent of ε). For (homogeneous and) isotropic plates of uniform thickness, they are both positive (so that μ_1 and μ_2 are real) if $v > 0.18535$. The form (35) is appropriate for P in this case. For smaller values of v, μ_1^2 and μ_2^2 are complex conjugates. In either range of v, the solution (35) satisfies the two boundary conditions (21a, b) at $x = 0$. The constants C_1 and C_2 are determined by the two conditions (22a) and (22b) at $x = 1$. For $0 < \varepsilon \ll 1$, the complementary component of (35) associated with C_1 and C_2 are evidently boundary layer solutions. This can also be seen from the singular perturbation structure of the ODE (20) given that $D_P\bar{C}_T/L^4 = 0(\varepsilon^4)$.

It is not difficult to see from the explicit solution for C_1 and C_2 (not given here) that for $v > 0$, we have

$$P(1) = 0(1), \quad P(0) = 0(e^{-\mu_1/\varepsilon}), \quad P'(1) = 0\left(\frac{\mu_1}{\varepsilon}\right). \tag{37}$$

Actual numerical results show that both $P'(1)$ and $P(0)$ are negative for $\varepsilon \ll 1$. It follows from the expression (26) for the center deflection w_{12} that the correction to the Kirchhoff solution by the twelfth order theory is $0(\varepsilon)$ for $v_z > 0$ since

$$12(1+v)\frac{D_MB_{MT}}{L^2}P'(1) = \frac{v_z\varepsilon}{1-v}\left[\frac{8\varepsilon P'(1)}{105}\right] = 0(\varepsilon). \tag{38}$$

Furthermore, this correction term is now seen to be a Poisson's ratio effect, associated with the transverse normal stress and consistent with the result given in (1a) for the interior solution of the three-dimensional elasticity problem. On the other hand, we have $(1+v)B_{MT}/B_Q = v_z/126$ so that the contribution of the $0(\varepsilon)$ term of (38) is numerically small compared to the $0(\varepsilon^2)$ correction of the sixth order theory except for $\varepsilon = 0(10^{-2})$ or smaller. This is consistent with the results of Gregory and Wan (1984) for the interior solution.

By the presence of the $0(\varepsilon)$ in the central deflection and the layer phenomena in the stress and displacement distributions, the twelfth order theory has succeeded in capturing additional qualitative features of the exact solution not possible by the lower order theory.

The actual numerical value of the center deflection w_{12} as a function of the thickness-to-width ratio is shown in Fig. 1 for $v = 1/3$. With the $P'(1)$ term dominating the $P(0)-P(1)$ term in (27) and with $P'(1) < 0$ for $0 < \varepsilon \ll 1$, w_{12} is smaller than w_R for $0 < \varepsilon \ll 1$. The graphs in Fig. 1 show that this persists for larger ε as we would expect intuitively. The agreement between w_{12} and w_I is within 4.7% for $\varepsilon \leqslant 0.5$ for $v = 1/3$ (and $v = 1/2$ as well). For comparison, the agreement between w_R and w_I for $\varepsilon = 1/2$ is within 7.3% for $v = 1/3$. Hence the sixth order theory is adequate even for relatively thick plates (with a thickness-to-span ratio up to 1/2).

9. THE SHEARED BLOCK PROBLEM

Another cylindrical bending problem analyzed in Gregory and Wan (1984) is the sheared rectangular block problem for the same infinitely long plate strip of Section (2). For the new problem, the top and bottom faces $x_3 = \pm h$ are freed from surface tractions so that

$$x_3 = \pm h: \quad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \quad (|x_2| < \infty, |x_1| \leqslant L). \tag{39}$$

The two sides $x_1 = \pm L$ of the plate strip are bonded to rigid walls which are displaced a distance $\pm w_0$ in the x_3-direction (uniformly in the x_2-direction). Of interest here is the transverse force (per unit edge length) $\pm Q_0$ needed at the edges to produce these edge displacements.

An approximate solution by Kirchhoff's plate theory is

$$Q_0 \simeq \frac{4G\varepsilon^3 w_0}{(1-v)} \equiv Q_K. \tag{40}$$

By Gregory and Wan's (1984) method, the correct solution (up to exponentially small terms), denoted by Q_I, is

$$\frac{Q_K}{Q_I} = 1 - \frac{3vt_2^B}{2(1-v)}\varepsilon + \frac{6-3vt_2^F+(2-v)n_3^B}{2(1-v)}\varepsilon^2 + \frac{(2-v)n_3^F}{2(1-v)}\varepsilon^3 \tag{41}$$

where $Q_I \cong Q_K/2$ for $\varepsilon = h/L = 0.5$ and $v = 0.5$.

It is straightforward to apply the sixth order plate theory for the same problem to obtain the mid plane transverse deflection w_R:

$$\frac{w_R}{w_K} = \frac{3/2}{1+3D_M B_Q/L^2}\left\{x - \frac{x^3}{3} + \frac{2D_M B_Q}{L^2}x\right\} \tag{42}$$

and the corresponding transverse force at the edge, denoted by Q_R:

$$\frac{Q_K}{Q_R} = 1 + \frac{3D_M B_Q}{L^2} = 1 + \frac{12\varepsilon^2}{5(1-v)}. \tag{43}$$

Similar to the case of uniform pressure loading, the sixth order plate theory solution for the sheared block problem does not contain any layer component; its first order correction to the Kirchhoff plate solution is $0(\varepsilon^2)$ (and not $0(\varepsilon)$ as in Q_K/Q_I). On the other hand, comparison of the numerical value of Q_R and Q_I for $0 \leqslant \varepsilon \leqslant 0.5$ shows remarkably

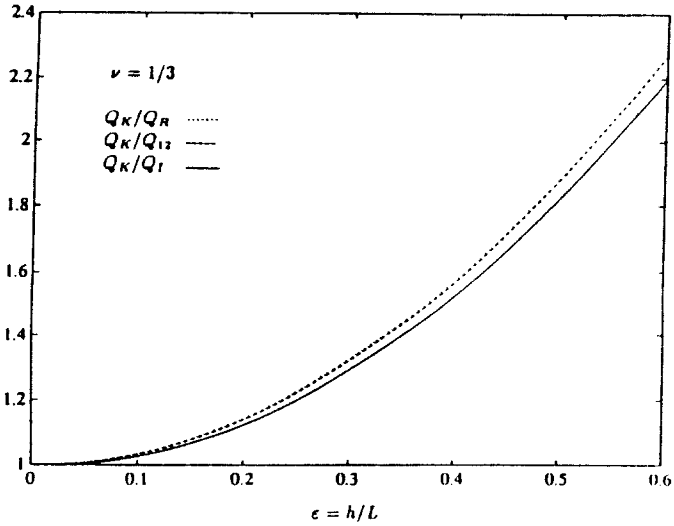

Fig. 2. The reciprocal of the end transverse shear resultants for $\nu = 1/3$ (normalized by the corresponding Kirchhoff theory solution) as functions of the thickness-to-span ratio $\varepsilon = h/L$ obtained from (i) the sixth order plate theory, (ii) the twelfth order plate theory, and (iii) the interior solution of Gregory and Wan.

good agreement (see Fig. 2). The discrepancy at $\varepsilon = 0.5$ is less than 4.1% of Q_I for $\nu = 1/3$. In contrast, the corresponding discrepancy for Kirchhoff's theory is 82%. Hence, the sixth order theory is again adequate for this problem.

A solution of the problem has also been obtained using the twelfth order theory with the edge conditions at $x = \pm 1$ being $\phi_1 = \psi_1 = v = 0$ and $w = \pm w_0$. The corresponding expression for the transverse force, denoted by Q_{12}, is

$$\frac{Q_K}{Q_{12}} = \frac{Q_K}{Q_R} - \frac{3(1+\nu)D_M B_{MT}}{L^2} P'(1) + \frac{3D_M}{L^2}[B_{QS} + (1+\nu)B_{MT}]P(1). \tag{44}$$

The dimensionless function $P(x) \equiv P_{11}/Q_0 L$ is determined by the fourth order ODE (20) with the right-hand member set to zero and the four boundary conditions

$$x = 0: \quad P = P'' = 0 \tag{45a, b}$$

$$x = 1: \quad \frac{\bar{C}_T}{L^2} P'' - B_{PT}(1+\nu)P = B_{MT}(1+\nu)x \tag{45c}$$

$$\frac{\bar{C}_T}{L^2} P''' - [B_{PT}(1+\nu) + B_S]P' = B_{QS} + B_{MT}(1+\nu). \tag{45d}$$

The exact solution of this BVP shows that $P(x)$ is an $0(1)$ layer solution with a layer width of order h and with $P'(1) > 0$. Hence the contribution of the $P'(1)$ term in (44) is $0(\varepsilon)$ and has the same sign as the corresponding term in Q_K/Q_I, just as in the uniform pressure load case. Quantitatively however, the presence of the additional terms in (44) associated with $P(1)$ and $P'(1)$ changes Q_K/Q_R by less than 2% for $\varepsilon \leqslant 0.6$ and $\nu = 1/3$ (or 1/2). The three ratios, Q_K/Q_R, Q_K/Q_{12} and Q_K/Q_I are plotted as functions of the thickness-to-span ratio ε in Fig. 2 for $\nu = 1/3$.

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